Mechanism for the Faraday instability in viscous liquids

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This report presents a discussion of the wave number selection mechanism in the Faraday instability, which arises when standing waves form on the free surface of a liquid subject to vertical sinusoidal oscillation. The focus is on the case where the viscous effects are strong, i.e., when the wavelength is of the same order of magnitude as the boundary layer thickness at the free surface. We investigate the relationship between the Faraday instability and the Rayleigh-Taylor instability by performing linear stability calculations. In the deep water limit (wavelength $\ll$ liquid depth), our results indicate that the preferred wave number in the Faraday instability is primarily determined through a Rayleigh-Taylor instability. In the case of shallow water (wavelength $\sim$ liquid depth), the agreement between the Rayleigh-Taylor and Faraday wave numbers does not appear to be as good, probably due to the interaction between the oscillatory motion of the standing waves and the bottom boundary.

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Since their observation by Faraday in 1831 [1], the standing waves produced on the free surface of a liquid undergoing vertical sinusoidal oscillation have been extensively studied. Much of that study has been done recently, as these waves provide a convenient way to investigate pattern formation, stability, and dynamics in spatially extended nonlinear systems (e.g., Refs. [2]– [5] and references therein). Depending on the properties of the liquid, the depth of the liquid layer, and the oscillation frequency, patterns as diverse as stripes, squares, and hexagons can be selected at instability onset. Of particular interest is the use of relatively viscous liquids (e.g., glycerol-water solutions [3], silicone and paraffin oils [4,6]), which damp sidewall effects and cause the selected pattern to be independent of container shape. This was first demonstrated by Edwards and Fauve, who used a two-frequency forcing scheme to generate surface waves possessing quasicrystalline order in containers of different shapes [3].

A measure of the influence of viscous forces on the Faraday instability is given by the product $k\delta$, where k is the instability wave number and δ is the boundary layer thickness at the free surface [7]. k^{-1} provides a measure not only of the instability wavelength, but also of the depth to which the free surface disturbance extends into the bulk liquid. An order of magnitude estimate of δ is $\sqrt{\nu/\omega}$, where ν is the kinematic viscosity of the liquid and ω is the forcing frequency. When $k\delta \ll 1$, viscous effects are weak, while $k\delta \sim 1$ implies that viscous effects are strong. The other important length scale in the problem is the liquid depth, h. $kh \gg 1$ is known as the deep water limit, while we will refer to $kh \sim 1$ as the shallow water case.

The behavior of the standing waves in each of the regimes suggested by the above characterization has previously been examined through linear stability analyses; we briefly summarize some key results. In the limit of deep water and weak viscous effects ($kh \gg 1, k\delta \ll 1$), the waves can be modeled by a damped Mathieu equation [7]:

$$\frac{d^2\zeta}{dt^2} + 4\nu k^2 \frac{d\zeta}{dt} + [gk + \sigma k^3/\rho - ak\cos(\omega t)]\zeta = 0. \quad (1)$$

Here, ζ is the surface deformation (amplitude of the surface waves after the spatial dependence has been factored out), σ is the surface tension, a is the amplitude of the forcing acceleration, and g is the acceleration of gravity. The physical model this equation corresponds to is that of a damped pendulum whose pivot point oscillates vertically at frequency ω and amplitude a [8]. The instability is parametric in nature since the forcing appears in the coefficients of (1) rather than through an inhomogeneous term. As first noted by Benjamin and Ursell, standing waves can form even if $a/g < 1$ [9]. When the water becomes shallow and viscous effects remain weak ($kh \sim 1$, $k\delta \ll 1$), an equation which is nonlocal in time (i.e., an integro-differential equation) is needed to describe the behavior of ζ [10,11]. The nonlocality arises because of viscous dissipation at the bottom boundary, and it corresponds to a history-dependent damping in the pendulum model mentioned above. Again, instability can occur even if $a/g < 1$.

The surface deformation also obeys a nonlocal equation when the water is deep and viscous effects are strong ($kh \gg 1, k\delta \sim 1$) [12]. However, Cerda and Tirapegui have shown that a local equation can quantitatively describe ζ when the water becomes shallow [7]. This equation is again a Mathieu equation, but the dependence of the damping on the wave number is much different than when viscous effects are weak. In this regime ($kh \sim 1, k\delta \sim 1$), diffusion of momentum occurs so rapidly relative to the surface oscillations that the damping no longer depends on the history of the motion. When viscous effects are strong, the critical acceleration amplitude, a_c, required to excite the standing waves is typically larger than the acceleration of gravity, g.

While a number of studies have explored the behavior of the Faraday instability when viscous effects are strong, less attention has been paid to the physical mechanisms respon-

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sible for this instability. Although the instability can still be characterized as parametric, the fact that $a_c/g>1$ suggests that a Rayleigh-Taylor (RT)-type mechanism may be important because the container experiences a body force in the direction opposite to gravity for part of the oscillation period. Lioubashevski *et al.* explored this idea by scaling the mean upward acceleration based on a_c with $h\omega^2$, and plotting the result against $(\delta/h)^2$ [13]. The data from a number of different experiments in the shallow water case were found to collapse onto a single curve, and the scaling was found to break down as a_c approached g. This issue was also addressed by Cerda and Tirapegui. Based on the model Mathieu equation they developed, they stated that ''...the amplification is due essentially to the fact that the system spends time in regions of effective negative accelerations which is an unstable situation as the Rayleigh-Taylor instability shows...'' (p. 221 of Ref. [7]).

Although both of the above studies recognize the potential importance of a RT instability, neither explicitly calculates and compares a RT wave number with a Faraday wave number. In this work, we bridge that gap by performing linear stability calculations for both the Faraday and RT instabilities. First, for a given set of problem parameters, we compute a_c along with the corresponding critical wave number, k_c. Then, using a mean upward acceleration based on a_c, we calculate the fastest growing wave number of the associated RT instability. Good agreement of this wave number with that from the Faraday instability would indicate that the wave number selection mechanism is a RT instability.

The calculations concerning the Faraday instability are performed by the method we used previously [14]. A recursion relation for the temporal modes of the free surface deformation is developed, converted into an eigenvalue problem, and then solved numerically. The calculations concerning the RT instability are performed by solving the relevant dispersion relations using Newton's method. In the infinite-depth case, we solve

$$-\overline{a_c}k+\frac{\sigma}{\rho}k^3+\nu^2(q^4+2q^2k^2-4qk^3+k^4)=0, \qquad (2)$$

while in the finite-depth case, we solve

$$-\overline{a_c}k+\frac{\sigma}{\rho}k^3-\nu^2\left(\frac{4k^2q(k^2+q^2)-C\cosh(qh)\cosh(kh)+D\sinh(qh)\sinh(kh)}{q\cosh(qh)\sinh(kh)-k\cosh(kh)\sinh(qh)}\right)=0, \qquad (3)$$

$$C=q(q^4+2q^2k^2+5k^4), \qquad (4)$$

$$D=k(q^4+6q^2k^2+k^4). \qquad (5)$$

Here $q^2=k^2+s/\nu$, where s is the instability growth rate. The mean upward acceleration, $\overline{a_c}$, is defined to be

$$\overline{a_c}=\frac{\omega}{\pi}\int_{3\pi/2\omega}^{5\pi/2\omega}[a_c\cos(\omega t)-g]dt=\frac{2}{\pi}a_c-g. \qquad (6)$$

These dispersion relations can be derived from those for unforced free surface waves [10,14]; one simply needs to replace g with $-\overline{a_c}$ and assume real-valued growth rates. Validation was carried out by comparing our infinite-depth results with those of Joseph *et al.* [15], and excellent quantitative agreement was achieved. As expected, our finite-depth results approach those for infinite depth as h increases. In our calculations, we fix $\sigma=20$ dyn/cm and $\rho=1$ g/cm^3 while varying ν. The value of ρ is characteristic of most liquids used in Faraday instability experiments, and the value of σ is typical of organic liquids. For water-based solutions, σ is closer to 70 dyn/cm, but this does not change the qualitative nature of our results as indicated by other calculations we have performed. The forcing frequency in the Faraday calculations is varied between 20 and 180 Hz, which is also typical of experiments. As the forcing frequency increases, so do a_c and k_c. We note that over the range of parameters examined in this study, the standing waves respond subharmonically to the forcing.

The infinite-depth case corresponds to the deep water limit and we discuss these results first. In Fig. 1, we plot the critical wave number from the Faraday and RT calculations versus a_c/g. For $\nu=1$ cm^2/s, the RT results underpredict the Faraday results by 25% (at $a_c/g\sim 58$) to 42% (at $a_c/g\sim 5$) [Fig. 1(a)]. Much better agreement is achieved when viscous effects become stronger, as seen in Fig. 1(b) where $\nu=10$ cm^2/s. Now, the RT results underpredict the Faraday results by 1% (at $a_c/g\sim 6$), and overpredict them by 6% (at $a_c/g\sim 203$). Finally, Fig. 1(c) demonstrates that the agreement continues to be reasonably good (within about 10%) for $\nu=100$ cm^2/s. These results indicate that for the case of deep water and strong viscous effects, the preferred wave number in the Faraday instability is primarily determined through a Rayleigh-Taylor instability.

To study the effects of shallow water, we take $h=0.5$ cm (which gives $kh\sim 1$) and $\nu=10$ cm^2. Now, the RT wave number overpredicts the Faraday wave number by 16% (at $a_c/g\sim 208$) to 141% (at $a_c/g\sim 32$). Thus, the agreement between the RT and Faraday wave numbers in shallow water does not appear to be as good as in deep water. We note two other observations about the finite-depth results. First, the tendency of the RT results to overpredict the Faraday results is primarily due to the increased sensitivity of the RT dispersion relation to the fluid depth. For example, when $\nu=10$ cm^2/s and the forcing frequency is 20 Hz, the critical wave numbers for the Faraday instability are 1.48 cm^{-1} at infinite depth and 1.93 cm^{-1} at $h=0.5$ cm. In contrast, the corresponding RT wave numbers are 1.47 and 4.65 cm^{-1}, respectively. Second, the poorer agreement of the RT and Faraday wave numbers at finite depth is not due to the fact that the maximum RT growth rate is smaller than the standing wave frequency. Again considering $\nu=10$ cm^2/s and a

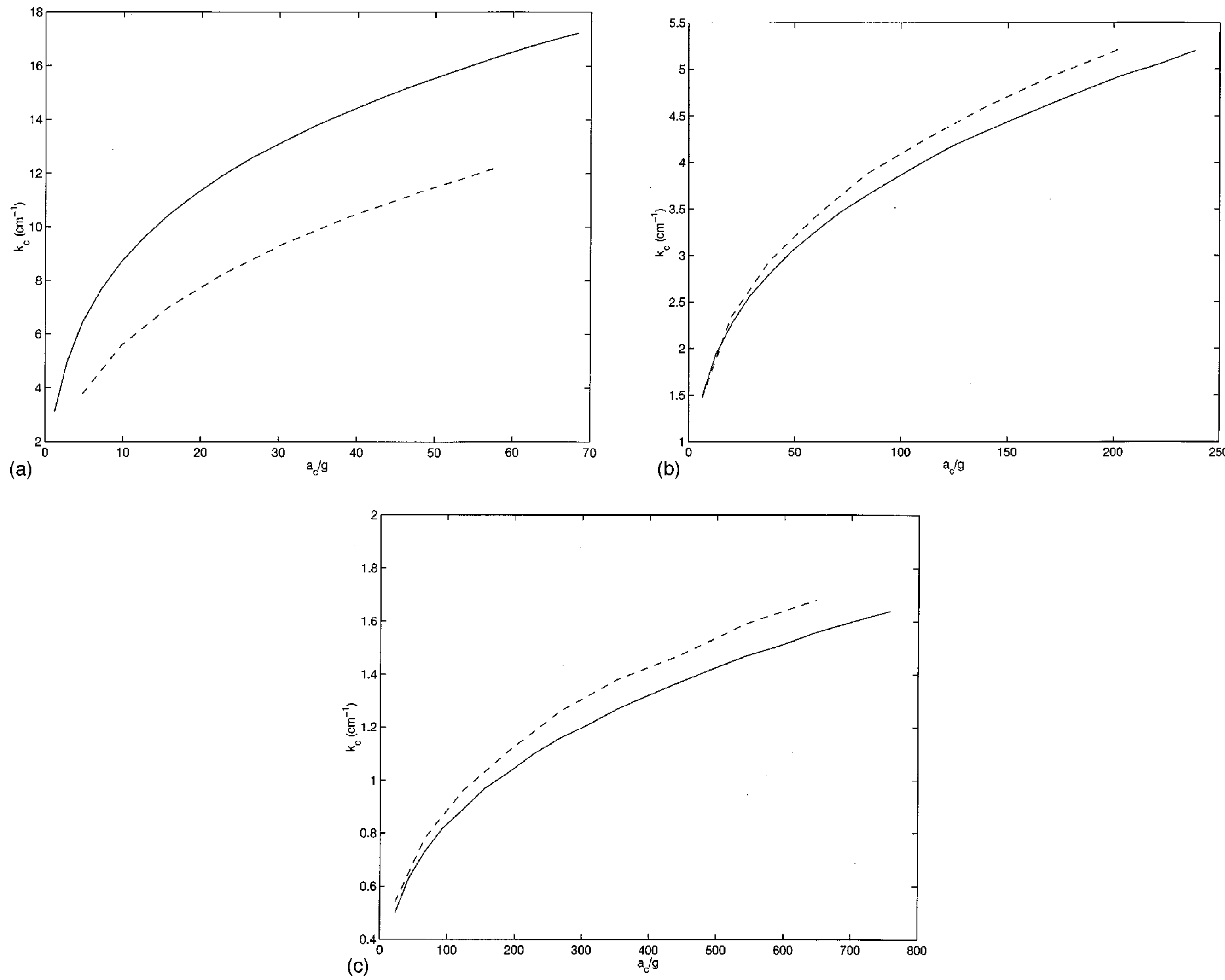

FIG. 1. Critical wave number at infinite depth for the Faraday instability (solid line) and fastest growing wave number for the corresponding Rayleigh-Taylor instability (dashed line) vs the critical acceleration amplitude, a_c: (a) $\nu=1$ cm^2/s, (b) $\nu=10$ cm^2/s, (c) $\nu=100$ cm^2/s.

forcing frequency of 20 Hz, the RT growth rate is 120 s^{-1} at $h=0.5$ cm, while the corresponding standing wave frequency is 62.8 s^{-1}.

The results in the shallow water case are perhaps not surprising, given that the standing waves undergo small amplitude oscillatory motion. This motion creates a flow which is not present in the RT instability, and can interact with the bottom boundary. One consequence of this flow seems to be the selection of longer wavelengths, as seen in the tendency of the RT calculations to overpredict k_c. As the fluid depth increases, the effect of this flow on the wavelength selection decreases. Our results should not be taken to mean that the RT instability does not play an important role in determining the preferred wave number for the Faraday instability when the water is shallow and viscous effects are strong. However, they do suggest that the oscillatory motion of the standing waves and the presence of the bottom boundary also need to be considered when elucidating the wavelength selection mechanism for this regime.

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